Technology Prompts New Understandings: The Case of Equality

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Changes to students' understanding of mathematical notation may be brought about by using technology within mathematics. Taking equality as a case study, the paper provides brief epistemological, historical, didactical, and computational reviews of its symbolic representation in pen-and-paper and technology-assisted mathematics, most especially in CAS. A multiplicity of special technology signs convey specific aspects of the broad meaning of the pen-and-paper sign. This provides a basis for new investigations into the effect on understanding of students' doing mathematics with technology.

Mathematical ideas are communicated through natural language, gestures, written symbols and diagrams. As students progress through schooling and are inducted into the mathematics community of practice, the use of symbols increases. At each stage the meaning of a range of symbols is "taken as shared" between the teacher and the students and between student and student and out to the wider mathematical world. Learning to work and communicate mathematically involves learning to correctly use and interpret these symbols and to harness their power.

This paper is motivated by observations that have arisen in our study of technology for use in mathematics classrooms (e.g., Pierce, Stacey, Wander & Ball, 2011), and sparked by one observation in particular. Looking at just a few computer algebra systems (CAS), we found seven signs that are used to convey meanings encompassed by the one familiar equals sign "=" of pen-and-paper mathematics. These signs may be 'written' (typed, pressed on a labelled button, displayed) by human or machine or 'read' by the machine when input or by the human user when output. By looking at more technologies, we know there are certainly more than 7 signs in use, but it is not the exact number that concerns us. Instead it is both the multiplicity of different signs and also the multiplicity of meanings encompassed by the familiar equals sign of pen-and-paper mathematics that is our interest.

=	==	===	:=	enter	EXE	→

Figure 1. Seven technology signs unwrapping meaning of the pen-and-paper equals sign.

The aim of this paper is to present an analysis of the different signs used in pen-and-paper mathematics contrasted with those of technology-assisted mathematics, as a first step towards long term investigation of whether the use of technology in the mathematics classroom affords opportunities for deeper discussion of the meaning of mathematical symbols. We have chosen equality as an initial case study, spurred on by the observation above. To study the equals sign, we especially look at computer algebra systems (CAS) which involve the symbolized equality most centrally. In the era where CAS are widely used by students, how might this fundamental sign, rich in meaning, be interpreted and

In V. Steinle, L. Ball & C. Bardini (Eds.), Mathematics education: Yesterday, today and tomorrow (Proceedings of the 36th annual conference of the Mathematics Education Research Group of Australasia). Melbourne, VIC: MERGA.

how might this affect students' understanding of this sign? Given space limitations, we offer brief epistemological, historical, didactical, and computational reviews of the symbolic representation of equality in pen-and-paper mathematics and in technology-assisted mathematics. We begin with pen-and-paper mathematics in the first sections, reflecting history of mathematics and the bulk of research into students' learning and then examine the issues arising from technology. Computer software in general (e.g., including spreadsheets or dynamic geometry) could have been investigated, but symbolic algebra already raises sufficient issues.

As this case study will show, providing instructions for a calculator or computer requires a level of precision greater than that in writing pen-and-paper mathematics for oneself or to communicate ideas to others. To communicate to a digital device, it is often necessary to make distinctions that are usually safely blurred in pen-and-paper mathematics or which can be explained with additional words. Hence we expect that to use technology well students may need deeper understanding of certain mathematical concepts and signs, and a lack of such understanding may cause problems for technology use. On the other hand, we know from studies of mathematics learning (there are examples later) that students' progress in pen-and-paper mathematics is often hindered when they do not understand the multiple shades of meaning that mathematical symbols can convey. Hence, it is possible that working in a technological environment, where clarifying meaning is essential, may sharpen students' awareness of some aspects of the meaning of mathematical symbols. We therefore expect that the observations in this paper will have relevance to learning and teaching both pen-and-paper mathematics and mathematics assisted by technology. The paper concludes with some open questions for new research.

Components of signs

This paper is based on an epistemological approach specific to mathematical notation by Serfati (2005). Notions from semiology and semiotics underlie the current study, such as the concepts of signifier and signified (although not referred to as such), syntax and semantics, but neither of these fields are utilised as frameworks *per se*.

Using a simplified version of Serfati (2005), what we see of a sign is its *materiality* (what it looks like, whether it is a letter or figure etc), and to use it we also need to know its *syntax* (how it combines with other signs), and *meaning*. To illustrate, consider the equation in Figure 2 and the familiar small dash "-" (let us simply call it the minus sign for now) which appears three times.

$$\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ b & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a-b & b \end{pmatrix}$$

Figure 2. Equation showing one sign used with three different meanings.

The materiality of this sign includes the straightness of the dash, its short length, and its position above the lower line of writing. In *a-b* the sign means subtraction of (unknown) numbers. In this context, the syntax includes that it is a binary operator on numbers, that the left/right order matters, and that in an expression such as "3×5–2", it does not take precedence. The minus sign in "-1" has the same materiality but a different meaning as it indicates a negative number. The syntax of this unary sign includes that it operates on the number to the right. In the third instance, the minus sign means subtraction of matrices, which is formally a different operation from subtraction of numbers with analogous but different syntax. The example shows that even within one short text, one sign may be used with several different meanings each requiring different syntax. To work with a sign, one not only has to recognise it in text (i.e., its materiality), but to select the right meaning and appropriate syntax in that context. As Figure 2 shows, the context sometimes has to be interpreted very locally (in front of a number, between matrices etc.). Meaning for Serfati is that commonly agreed by the community of mathematicians –it does not refer to a person's individual understanding, which may (for example) be erroneous.

The sign for equality in standard school mathematics, "=", also has multiple meanings, discussed below. Its materiality includes that there are two dashes (straight line segments), its short length, and its position centred above the lower line of writing. A key part of its syntax is that it always creates two "spaces" one on the right and one on the left. Its meaning is that it makes a statement concerning the complete mathematical objects in these two spaces. As we will see, technology-assisted mathematics makes explicit the range of different meanings encompassed by the single pen-and-paper equals sign.

Equality in pen-and-paper mathematics

History meets Didactics

In the realm of pen-and-paper mathematics, there is now one (material) sign "=" for equality. This sign has multiple meanings. The history of the symbols for equality is one pointer to this. For many centuries the representation for equality was restricted to natural language (words translating into English as make, equals etc). Equality was given a sign (an elongated version of the present day "=") in Robert Recorde's 1557 book *The Whestone of Witte*. Recorde selected his sign to indicate symmetry and sameness: "nothing could be more equal than two straight lines parallel to the writing line". Serfati (2005) notes that this sign was not immediately adopted. Descartes often used a loop, ∞ .,which conveyed an asymmetrical aspect in its materiality (Puig & Rojano, 2004). There would usually be a complicated expression on one side of the loop and a simplification or the quantity to be found (e.g., z) on the other. Figure 3 contains two sample equations from the section of Descartes's book where he derives geometrically the positive solution (RHS of Figure 3) to quadratic equations of the form $z^2 = az + b^2$ (a > 0).

Figure 3. Loop shaped sign for equality used by Descartes (La Géometrie, 1637, p.301).

A large body of mathematics education research has reported observations that learners of mathematics, whose education naturally begins in arithmetic, understand the equals sign "=" as an instruction to calculate an answer and an indication of where this answer should be placed. This is like Descartes' loop. Difficulties emerge, especially in learning algebra, when this 'evaluate' meaning of equals is the only meaning that students know. Many studies review this literature and report findings (e.g., Jones & Pratt, 2006; Kieran, 1981; Prediger, 2010; Matthews, Rittle-Johnson, McEldoon, & Taylor, 2012).

Sáenz-Ludlow and Walgamuth (1998) provides an example from research at grade 3. The teacher asked the students which number will make the number sentence "246 + 14 = ? + 246" true. As noted in this and other research, students with only an 'evaluate' understanding of the equals sign will often say 260, or sometimes 506 if they continue to add the final 246. When the teacher asked a student to explain the meaning of the equals sign, Student Sh said "*It's when you add something. The equal sign is there so you can put the answer by the equal sign*." This is a clear description of the evaluate meaning and its expression-equals-answer syntax. Later, the teacher works on another aspect related to equality: the validity of the written proposition "6 + 6 = 6 + 6". In this number sentence, the equals sign is not being used with the "expression-equals-answer" evaluate meaning, but to state a relationship. Student Sh said 6 + 6 = 6 + 6 was not true "... *because six plus six equals twelve, not six plus six.*" Another student Ka disagreed, commenting "*Yeah, it does because both of them equal the same amount. That could be real. You could do that.*" When the teacher emphasised that both sides were equal to 12, writing 12 under each side, student Sh still commented "*No, I don't get it.*" Other research (see, for example, the references above) demonstrates that Sh is also likely to have the same problem even with more "informative" statements such as "6 + 6 = 7 + 5". This common student difficulty mirrors the historical record which shows that the asymmetric character of equality that typified rhetorical mathematics took a long time to fade away in the symbolic register.

In algebra, and in the more algebraic aspects of studying number and writing expressions, students need both this 'evaluate' meaning and also the 'equate' meaning of the sign which states that two expressions (long or short, even one character) have the same value. For example, students can solve an equation such as $2x+3=11$ with understanding, if they know only the evaluate meaning of equality. When one multiplies the unknown number by 2 and adds 3, the answer is 11. So the answer to multiplying the number by 2 is (11-3)=8, so the unknown number is 4. However, to solve the equation $2x+3=x+7$, a different interpretation of the equality is required. This equation is on the further side of Filloy and Rojano's (1989) *didactic cut*: it is known that these equations are conceptually and practically more difficult. Instead of thinking about the answer to calculations, to solve this equation, we need to operate on it as a statement with a truth value. The logic is that if $2x+3=x+7$ is true, then $x+3=7$ is a true statement and so $x=4$ is a true statement. This involves the equate meaning of the sign and not the evaluate meaning. Jones and Pratt (2006), after a detailed literature review, comment that younger students may actually know both meanings but put the equate meaning aside because they do not find it useful in the mathematical work that they do. Godfrey and Thomas (2008) cite a number of other studies to show that the equate meaning of equality is quite amenable to instruction. They estimated that about 25% of their beginning tertiary students had not developed the firm understanding of equality as an equivalence relation (symmetric, reflexive, transitive) that underlies the equate meaning.

The meanings of equality

As noted above, mathematics education research draws attention to two different aspects of the meaning of equality, and the problems for teaching when students have developed only the 'evaluate' meaning when they need the 'equate' meaning. Prediger (2010), however, classifies the meanings of the equals sign in pen-and-paper mathematics, into three broad groups, labelled *operational meaning* (evaluate), *relational meaning* (equate) and *specification* (assign), when the equals sign is used to name. Although adopting Prediger's groupings, we choose different labels (in brackets), which better fit technology discussions. Terminology in the literature is diverse (e.g., operational is often called *procedural* or *arithmetic*, relational is often called *structural* or *algebraic* etc.). Table 1 summarises these three meanings of the one material "=" sign with examples based on Prediger (2010). These show that each of the three categories includes variations of the meanings (e.g., identity or conditional equations), but the three broad groups provide a useful classification for our purposes. The three meanings have different syntax. For example, whereas the order is very important for the evaluate and assign meanings, for the equate meaning, we can write "expression 1" = "expression 2" or vice versa.

We should note that all of the 'evaluate' examples can be read with a 'equate' meaning by someone familiar with both: the difference lies in the writer's or reader's intention in the context. So for example, a teacher may write "$2(x+1) =$" with the intention of having students expand the brackets and write the answer in the indicated space, producing "$2(x+1)=2x+2$". Students and the teacher may see this as giving an answer to the expansion (the evaluate meaning of "="). In writing this complete equation, they may or may not be stating that these two expressions are equal in the full equate meaning. In pen-and-paper mathematics, words are often used in conjunction with the symbols to supplement the meaning: "let $m = a + b$", "solve $\sin(x) = x$", "$2(x+1)=2x+2$ for all values of x". In technology-assisted mathematics, where words are used, they are (part of) the command (e.g., "Define $f(x) = 2x + 3$" in TI-Nspire) rather than an explanation.

Table 1
Three broad meanings of equals sign "=", after Prediger (2010)

Broad Meaning	Prediger's terminology	Syntax	Examples
Evaluate	Operational	expression = answer	$246 + 14 = 260$; $2(x+1) = 2x + 2$
Equate	Relational	expression 1 = expression 2	$246 + 14 = 14 + 246$, $260 = 246 + 14$ Solve $2x + 3 = x - 1$ $2(x+1) = 2x + 2$ $C = 2\pi r$
Assign	Specification	new name = expression	Let $m = a + b$

Equality and the equals sign in technology-assisted mathematics

There have been numerous reports in the research literature of the difficulty that many students have of adding the equate meaning of the equals sign to the earlier developed evaluate meaning. However, we know of no research which shows any problematic aspects of the assign meaning. Possibly in pen-and-paper mathematics, 'evaluate' and 'assign' equals are easy, and only 'equate' equals causes notable problems. The situation is different when using technology, both when writing (i.e., typing or pressing buttons) to communicate to the machine and when reading the output. We will demonstrate below that while the 'evaluate' meaning remains straightforward, the 'equate' and 'assign' meanings can both become very challenging.

Evaluate meaning in CAS

In CAS, the standard equals sign "=" is sometimes used in contexts where we might use the word "equals", sometimes other signs are used and sometimes no sign at all is used. Navigating through this field is part of the difficulty encountered by students in using CAS (see Guin, Ruthven & Trouche, 2005; Tonnison, 2011). In a typical interactive session the user input an expression *E* and the CAS evaluates it to a new form *R* that is output and may be considered the answer to the 'question' posed. Evaluation is triggered (and hence symbolised for action) by a key labelled "Enter" or "EXE" or even (especially in basic calculators) with the "=" sign. In this instance, all of these signs on the keyboard have the evaluate meaning of equality, so they were listed in Figure 1. In most cases, if *E* evaluates to *R* this implies that *E*=*R*, but no system known to the authors displays an sign signifying any sense of equals between input *E* and equivalent output *R*. Rows 1 and 2 in Figure 4 provide examples where such a sign could be used, but rows 3 and 4 show why it is not generalizable. An equals sign would also be inappropriate for the output after commands like changing the length of decimal display.

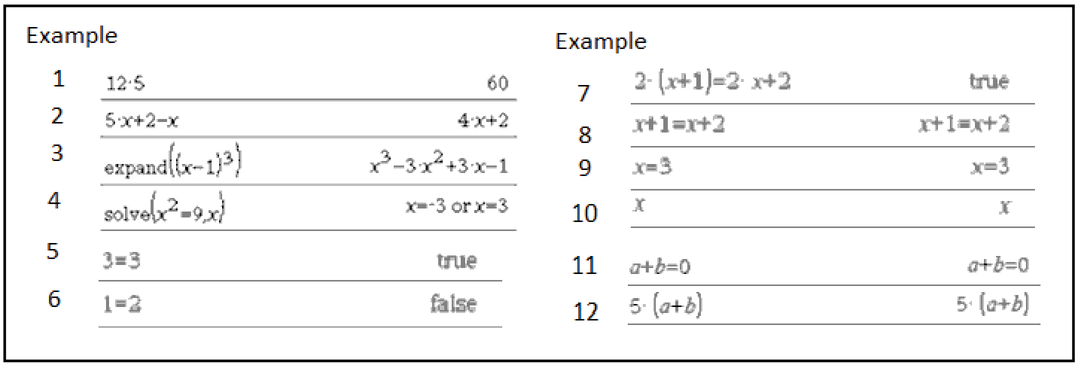

Figure 4. Screenshots from TI-Nspire.

Assign meaning in CAS

Variables play a double role: they can either stand for themselves (pure symbols, as *x* is in examples 2 and 3 of Figure 4) or refer to some other object. So, for example, the symbol *m* can be used for *a*+*b*, or we can define the function $f(x)=2x+1$. During evaluation of an expression, all variables that refer to some other object are replaced by that object. For example, after the command "*v*:=*3*" the number 3 is used wherever *v* is indicated. Variants of the sign ":=" include "STO" (store) and "→". Previous values or properties before the latest assignments are lost. Using assignment thus requires the students to pay attention to the order in which calculations and assignments take place. Complexity in using CAS ensues.

The sequence of TI-Nspire instructions "*a*:=3; *b*:=*a*+1; *b*" yields the value 4 for *b*. However, the instructions "*a*:=3; *b*:=*a*+1; *a*:=10; *b*" also yields the value 4 for *b*, although students might expect the answer to be 11. This is because the sign ":=" evaluates the left hand side at the time of evaluation of ":=", so when the instruction "*b*:=*a*+1" is issued, the evaluation at the time produces "*b* = 4". Changing *a* after this does not cause *b* to be recalculated. In some CAS (e.g., Mathematica) there is another symbol to signal delayed assignment so that evaluation takes place when the variable is referenced, not when the definition is given, providing the answer 11 in the case above. Unfortunately, the symbol use differs confusingly between the various systems. In TI-Nspire CAS, immediate assignment is ":=" but in Mathematica, "=" signifies immediate assignment and ":=" signifies delayed assignment. Substitution is another form of assignment: in this case, temporary assignment valid only for the evaluation of the expression. For substitution, TI-Nspire uses an ordinary "=" sign, whereas Mathematica uses an arrow. To substitute x=3 into $f(x)=2x+1$, in TI-Nspire the command is "$f(x) \mid x= 3$" and in Mathematica it is "$f(x)$ /. $x\rightarrow 3$".

The brief examples above show that using equality in its assign meaning is a complex aspect of using CAS, which has no parallel in pen-and-paper mathematics. Many of the additional technology signs for equals come from the various assign meanings.

Equate meaning in CAS

We now turn to the remaining category of the meaning of the equals sign –the 'equate' meaning where the central idea is that the mathematical objects on both sides of the equals sign are identical being in some equivalence relation, with the meaning of identical depending on context. This is denoted by "=" in TI-Nspire CAS, Maple and Maxima and by "= =" in Mathematica and Axiom. In a few CAS commands, the sign "=" is used as in pen-and-paper mathematics. For example, in "solve ($2*x=8$, x)", the "=" sign is only providing information about the typed equation to the solve command. However, most other instances of this type of equality in CAS are not so familiar to pen-and-paper mathematicians.

Calculators and computers are mostly used, as their name implies, to carry out routine procedures, especially complex ones. On the other hand, "=" when used in its equate meanings conveys knowledge –it states a relationship. For a mathematician it is interesting and important to know $A = \pi r^2$ and $-(a-b) = b-a$, but a calculator or computer only needs to 'know' these in so far as they assist with calculating and computing. As Serfati (2005) notes, however complex an operational instruction can be, it will never constitute, mathematically speaking, the ultimate goal. Adding five to two or calculating an infinite sum will only be intermediary steps undertaken to reach a different goal such as the proof of a universal property or the value of an unknown quantity. However a statement of equality is, in mathematics, semantically complete and can constitute the ultimate goal.

In CAS, inputs are best thought of as commands to do something, not to make a statement. Consider examples 5 to 12 of Figure 4, where entering into TI-Nspire a statement of equality has produced "true", "false" and some algebraic expressions. The explanation is that in TI-Nspire, "=" is the command to test for identity, whether one side is always equal to the other. In examples 5, 6, and 7, this is immediately decided, and so the outputs of true or false are given. When identity cannot be immediately decided as in examples 8, 9 and 11, the output is just the same as the input. A surprising result is that the statement in example 8 is not immediately identified as false. Instead, just the unevaluated equation is returned. Space precludes an explanation of why this happens. However, from personal experience we know that behaviour such as this puzzles novice CAS users.

For students, a surprising and probably disappointing feature of CAS is that inputting a statement of equality such as "$x = 3$" or "$a+b = 0$" does not make these statements true, as can be seen by comparing example 9 with the immediately following request for x in example 10 and comparing example 11 with example 12. This contrasts with pen-and-paper mathematics, where after writing "$a+b = 0$", the writer would use this information. For CAS, the input "$a+b = 0$" is only a command to test identity, not a command to make it true. Different steps need to be taken to force $a+b = 0$ in subsequent CAS calculations.

Conclusion

In the sections above, we have described the most common signs and meanings for equality. CAS also have other "equality" commands and their associated signs and syntax (e.g., whether two objects have the same address in computer memory or whether they have the same internal representation). It is clear, however, from the above that digital technology has "widened the use-meanings of the equals sign beyond those afforded by static media." (Jones and Pratt, 2006, p. 302). The case study of equality illustrates that communicating with technology presents new challenges for symbolization processes, and also that pen-and-paper mathematical language and CAS language need a n on-trivial translation. It also demonstrates that using technology to do mathematics requires a deeper understanding of some mathematical ideas than doing pen-and-paper mathematics.

As noted earlier, this analysis has been motivated by an interest in whether the use of technology in the mathematics classroom affords opportunities for deeper discussion of the meaning of mathematical symbols (including in pen-and-paper mathematics), and how building awareness of the multiple meanings hidden within symbols might work in practice. We expect that the following research questions may be fruitful. Does the case study of equality represent a t ypical or an unusual case? Does using technology affect students' pen-and-paper mathematics understandings of symbols (for example, does it help students grasp the different meanings behind the single paper-and-pencil equals sign)? Do the distinctions between meanings of signs that need to be made in communicating with technology point to important obstacles in learning mathematics that are not yet explored? Will new pen-and-paper notations develop as people work regularly in both environments? The above are questions from the student perspective, but related questions from the teachers' and teaching perspective are also important.

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